

INDECOMPOSABLE CONTINUA IN EXPONENTIAL DYNAMICS-HAUSDORFF DIMENSION

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ABSTRACT. We study some forward invariant sets appearing in the dynamics of the exponential family. We prove that the Hausdorff dimension of the sets under consideration is not larger than 1. This allows us to prove, as a consequence, a result for some dynamically defined indecomposable continua which appear in the dynamics of the exponential family. We prove that the Hausdorff dimension of these continua is equal to one.

1. INTRODUCTION

The dynamics of transcendental meromorphic functions has been developed as a counterpart to the theory of rational iterations. A model family $\lambda \mapsto \lambda \exp(z)$ has been a subject of a particular interest, playing a similar role as the family of quadratic polynomials $z^2 + c$ in the iterations of rational maps, see e.g. [De] for a review of results on this family.

There are several properties of the iterations of entire (and, more general, transcendental) maps which have no counterpart in the dynamics of rational maps.

Below, we recall basic definitions and facts.

Let $f : \mathbb{C} \rightarrow \hat{\mathbb{C}}$ be a transcendental meromorphic function. As usually, we shall denote by f^n the n -th iterate of f : $f^n = f \circ f \circ \dots \circ f$. The sequence $f^n(z)_{n=0}^\infty$ is called the trajectory of z . The Fatou set $F(f)$ consists of all points $z \in \mathbb{C}$ for which there exists a neighbourhood $U \ni z$ such that the family of iterates f_U^n is defined in U and forms a normal family. The complement of $F(f)$ is called the Julia set of f and it is denoted by $J(f)$. An intuitive characterisation of the Julia set says that it carries the chaotic part of the dynamics. See e.g. [Be] and [KU] for a detailed presentation of the theory.

For the particular family of maps $f_\lambda(z) = \lambda e^z$ the structure of the Julia set and the dynamics have been studied intensively. In 1981 M. Misiurewicz answered the old question (formulated by Fatou) and proved that the Julia set of the map $f_1(z) = e^z$ is the whole plane ([Mi]). A striking result, proved independently by M. Lyubich [Lyu] and M. Rees [Re] says, that, nevertheless, the map is not ergodic with respect to the two-dimensional Lebesgue measure, and for Lebesgue almost every point z the ω -limit set of z (i.e. the set of accumulation points of the trajectory) is just the trajectory of the singular value 0 plus the point at infinity.

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On the other hand, there is an open set of parameters λ for which there exists a periodic attracting orbit (i.e. a point p_λ such that $f_\lambda^k(p_\lambda) = p_\lambda$ and $|(f_\lambda^k)'(p_\lambda)| < 1$ for some $k \in \mathbb{N}$). In this case, the Fatou set is nonempty and it contains a neighbourhood of this periodic attracting orbit. Actually, it turns out that in this case the Fatou set is just the basin of attraction of this periodic orbit, i.e.

$$F(f_\lambda) = \{z \in \mathbb{C} : \lim_{n \rightarrow \infty} f_\lambda^{nk}(z) = f^i(p_\lambda) \text{ for some } i \in \{0, 1, \dots, k-1\}\}.$$

Moreover, this basin of attraction must contain the whole trajectory of the singular value 0. For instance, if $\lambda \in (0, \frac{1}{e})$, then f_λ has a real attracting fixed point p_λ . For these values of λ the Fatou set of f_λ is connected and simply connected and its complement $J(f_\lambda)$ is a "Cantor bouquet" of curves. See [De] or [Sch] for a survey of results in this direction.

A general classification theorem for the components of the Fatou set leads to the following corollary: if the parameter λ is chosen so that $f_\lambda^n(0) \rightarrow \infty$ then the Julia set of f_λ is the whole plane: $J(f_\lambda) = \mathbb{C}$ (see [EL] or [GK]).

In this paper, we shall always assume that the parameter λ is chosen so that $f_\lambda^n(0) \rightarrow \infty$ sufficiently fast. Thus, in particular, for all maps f_λ considered in this paper we have $J(f_\lambda) = \mathbb{C}$ (see Section 2 for setting of precise assumptions).

We shall explain now the motivation of the present paper.

It is known that the set of escaping points

$$I(f_\lambda) = \{z : f_\lambda^n(z) \rightarrow \infty\}$$

can be described in terms of so-called "dynamic rays" (see, [SchZi] and [Rem2]). Each dynamic ray is a curve $g_{\underline{s}} : (t_{\underline{s}}, \infty) \rightarrow \mathbb{C}$. Moreover, $\operatorname{Re}(g_{\underline{s}}(t)) \rightarrow +\infty$ as $t \rightarrow +\infty$. Each such curve is characterised by an infinite sequence $\underline{s} = (s_0, s_1, \dots)$ of integers. The sequence \underline{s} is frequently called the "external address" of the curve $g_{\underline{s}}$. The dynamical meaning of the sequence \underline{s} is the following: Let us divide the plane \mathbb{C} into horizontal strips

$$(1.1) \quad P_k = \{z \in \mathbb{C} : (2k-1)\pi - \operatorname{Arg}(\lambda) < \operatorname{Im}(z) \leq (2k+1)\pi - \operatorname{Arg}(\lambda)\}$$

If z is a point on the curve $g_{\underline{s}}$, $z = g_{\underline{s}}(t)$ with t sufficiently large, then, for every $n \geq 0$, $f_\lambda^n(z) \in P_{s_n}$.

The classification of escaping points (see [SchZi], Corollary 6.9) says that this family of curves almost exhausts the set $I(f_\lambda)$. Namely, if $z \in I(f_\lambda)$ then either z belongs to some curve $g_{\underline{s}}$, or z is a landing point of some curve $g_{\underline{s}}$, or else the singular value 0 escapes, $0 = g_{\underline{s}}(t_0)$ for some $t_0 > t_{\underline{s}}$, and z is eventually mapped to the initial piece of the curve $g_{\underline{s}}$, cut off by the point 0, i.e.

$$f_\lambda^n(z) = g_{\underline{s}}(t')$$

for some $t_{\underline{s}} < t' < t_0$.

As an example, let us consider $\lambda = 1$ and the dynamical ray corresponding to the sequence $\underline{s} = (0, 0, 0, \dots)$. This set is just the real line \mathbb{R} . The set

$$\bigcup_{n \geq 0} f_{\lambda}^{-n}(\mathbb{R}) \cap \{\operatorname{Im}(z) \in [0, \pi]\}$$

is a union of infinitely many arcs extending to infinity. It was observed first by R. Devaney (see [De]) that the closure of this set, after a natural compactification at ∞ , becomes an indecomposable continuum. This continuum can be equivalently described as the set of points whose trajectories remain in the strip $\operatorname{Im}(z) \in [0, \pi]$.

Next, again for $\lambda = 1$, R. Devaney and X. Jarque discovered the existence of indecomposable continua defined as an accumulation set, in the Riemann sphere, of some dynamic rays. More precisely, they considered in [DJ] the rays of the form $g_{\underline{s}}$, where $\underline{s} = (t_{m_1}, 0_{n_1}, t_{m_2}, 0_{n_2} \dots)$, where t_{m_j} are blocks of integers, of length m_j , with all digits $\leq M$. If the blocks of zeroes 0_{n_j} are sufficiently long, then the set of accumulation points of the ray $g_{\underline{s}}$ becomes an indecomposable continuum containing $g_{\underline{s}}$.

A more general result appeared in [Rem]. The author shows the following: assume that the singular value 0 of the map f_{λ} is on a dynamic ray, or it is a landing point of such a ray. Then there are uncountably many dynamic rays g whose accumulation set (in the Riemann sphere) is an indecomposable continuum containing g .

The present paper was motivated by the abovementioned examples. Our goal was to give a bound for the Hausdorff dimension of dynamically defined indecomposable continua appearing in the exponential dynamics. It was already observed in [De2] that M. Lyubich's result [Lyu] implies that two-dimensional Lebesgue measure of the indecomposable continuum described in [De], is equal to 0. Actually, it is easy to see that the same remark applies to the examples considered in [DJ].

Our goal in this paper is to prove, for a class of dynamically defined indecomposable continua for the exponential family, that their Hausdorff dimension is equal to one, so it takes on the smallest possible value. This class of continua contains, in particular, the examples described in [De] and [DJ]. So, our result is an essential strengthening of the previously known estimates.

Our results apply also to a subclass of the class of continua described in [Rem]. It is now natural to ask about the possible values of the Hausdorff dimension of other dynamically defined indecomposable continua (appearing in the dynamics of exponential maps), not covered by our considerations (in particular- all the continua described in [Rem]).

Our paper is organised as follows. In Section 2 we formulate the condition on the parameter λ ("super-growing parameters", see Definition 1). Then we introduce and describe some forward invariant sets for the dynamics of the map f_{λ} , where λ is a super-growing parameter.

We formulate a result (Theorem 4) on the upper bound of the Hausdorff dimension of these forward invariant sets, namely, that the Hausdorff dimension of such set is not larger than 1. This is a result of independent interest, useful for further applications.

The proof of Theorem 4 is divided into several steps, and it is presented in Sections 3, 4, 5 and 6.

In Section 7 we show in detail how Theorem 4 can be used to estimate the dimension of the particular, dynamically defined indecomposable continua. In each case when the application of Theorem 4 is possible, we get the strongest possible upper bound on the dimension, proving that the Hausdorff dimension of the continuum is equal to one. See Theorems 17 and 19 for the precise statement. In addition, it is worth noting that one-dimensional Hausdorff measure of an indecomposable continuum in the plane is not σ -finite, see Proposition 21, Section 8.

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2. STATEMENT OF THE TECHNICAL RESULT

Throughout the paper we will work with the function $f_\lambda(z) = \lambda e^z$ for which the trajectory of the singular value 0 tends to infinity exponentially fast. We keep the following definition and notation, introduced in [UZ]. Let

$$(2.1) \quad \beta_n = f_\lambda^n(0), \quad \alpha_n = \operatorname{Re} \beta_n$$

Definition 1. We say that the parameter λ is *super-growing* if $\alpha_n \rightarrow \infty$ and there exists a constant $c > 0$ such that, for all n large enough,

$$(2.2) \quad \alpha_{n+1} \geq c e^{\alpha_n}$$

Note that the above condition is equivalent to

$$(2.3) \quad |\beta_{n+1}| \geq |\lambda| \exp\left(\frac{c}{|\lambda|} |\beta_n|\right)$$

and to

$$(2.4) \quad \alpha_n \geq \frac{c}{|\lambda|} |\beta_n|$$

From now on, unless stated otherwise, we assume that the parameter λ satisfies the super-growing condition (2.2), with the constant c . We shall keep the notation (2.1). To simplify the notation, we write $f(z)$ instead of $f_\lambda(z)$, if it does not lead to a confusion.

Remark. For example, the condition (2.2) is satisfied if $|\operatorname{Im}(\beta_n)|$ is bounded and $f_\lambda^n(0) \rightarrow \infty$ as $n \rightarrow +\infty$. Moreover, it was proved in [We] that the Hausdorff dimension of parameters λ satisfying the super-growing condition is equal to two. (See also [FS] for a detailed description of the structure of parameters λ for which the trajectory of the singular value escapes to ∞ .)

In order to formulate the main technical result, we introduce the following definition.

Definition 2. Let $W \subset \mathbb{C}$ be a closed set. Denote by $W(R)_+$ the intersection $W(R)_+ = W \cap \{\operatorname{Re} z = R\}$ and, analogously, $W(R)_- = W \cap \{\operatorname{Re} z = -R\}$. Let

$$w(R) = \max(\operatorname{diam} W(R)_+, \operatorname{diam} W(R)_-).$$

We call the set W *thin* if there is a constant $K > 0$ such that

$$(2.5) \quad \frac{|z|}{|\operatorname{Re}(z)| + 1} < K$$

for all $z \in W$, and

$$(2.6) \quad \lim_{R \rightarrow \infty} \frac{\log_+ w(R)}{\log R} = 0.$$

Remark 1. Obviously, every horizontal strip

$$\{z \in \mathbb{C} : |\operatorname{Im}(z)| \leq P\}$$

is a thin set.

Note also that the condition (2.5) implies that there is a cone symmetric with respect to the real axis, with the opening angle smaller than π and $R_0 > 0$ such that the intersection $W \cap \{|\operatorname{Re}(z)| \geq R_0\}$ is contained in this cone.

The following definition introduces the set which will be an object of our estimates.

Definition 3. Let $\lambda \in \mathbb{C} \setminus \{0\}$. Put $f = f_\lambda$. Let W be a thin set. We define the set Λ_W as

$$\Lambda_W = \{z \in \mathbb{C} : f^n(z) \in W \text{ for every } n \geq 0\}.$$

Note that the set Λ_W is a forward invariant closed subset of \mathbb{C} . Forward invariant means that $f(\Lambda_W) \subset \Lambda_W$.

Remark 2. Since we do not assume any dynamical condition on W , it may even happen that the set Λ_W is empty. However, in all our applications (see Section 7) the set Λ_W contains a non-trivial continuum, so it has Hausdorff dimension at least one.

We shall prove the following.

Theorem 4. *Let λ be a super-growing parameter. Let W be a thin set. Put $f = f_\lambda$ and let Λ_W be the set defined in Definition 3.*

Put

$$(2.7) \quad Y_M = \{z \in \mathbb{C} : |\operatorname{Re}(z)| \geq M\}$$

Then

$$(2.8) \quad \lim_{M \rightarrow \infty} \dim_H(\Lambda_W \cap Y_M) \leq 1.$$

(Note that the limit exists because the function $M \mapsto \dim_H(\Lambda_W \cap Y_M)$ is non-increasing).

So, writing Λ_W as a union $\Lambda_W = \Lambda_{W,bd} \cup \Lambda_{W,ubd}$, where $\Lambda_{W,bd}$, $\Lambda_{W,ubd}$ denote the subset of points in Λ_W with bounded (resp.: unbounded) trajectory, we have

Corollary 5.

$$\dim_H(\Lambda_{W,ubd}) \leq 1.$$

Proof. Obviously, for every $M > 0$,

$$\Lambda_{W,ubd} \subset \bigcup_{n=0}^{\infty} f^{-n}(\Lambda_W \cap Y_M).$$

Since f is an analytic non-constant map, every Borel set A has the same Hausdorff dimension as its preimage $f^{-1}(A)$. Taking a countable union of sets of the same dimension does not increase the dimension. So, Corollary follows immediately from Theorem 4. \square

The proof of Theorem 4 is split into several steps. In Section 3 we claim the existence of the special induced map, and we formulate, in Proposition 6, some numerical estimates for this map. Next, the precise definition of the map and the proof of the required estimates is presented in Section 4 and 5. In Section 6 we use these estimates to conclude the proof of Theorem 4.

3. THE INDUCED MAP

Obviously, to prove Theorem 4 it is enough to consider integer values of M . So, from now on we shall assume that $M \in \mathbb{N}$.

In this section, we claim the existence of an auxiliary induced map, with certain properties, see Proposition 6 below.

First, we cut each horizontal strip P_k (see (1.1)) into rectangles

$$(3.1) \quad R_r^k = \{z : r \leq \operatorname{Re}(z) < r + 1\} \cap P_k$$

For an arbitrary $M \in \mathbb{N}$ denote by Z_M the family of all rectangles R_r^k intersecting $W \cap Y_M$.

Note that our assumption (2.6) on W implies that the number $n(r)$ of rectangles in Z_M intersecting the lines $\operatorname{Re}(z) = \pm r$, satisfies

$$\log n(r) = o(\log(r)).$$

The map F will be defined in the union W_M of all rectangles intersecting the set $W \cap Y_M$:

$$W_M = \bigcup_{R_r^k \in Z_M} R_r^k.$$

The set W_M splits naturally into two subsets:

$$W_M^+ = W_M \cap \{z : \operatorname{Re}(z) > 0\},$$

$$W_M^- = W_M \cap \{z : \operatorname{Re}(z) < 0\}.$$

The next Proposition summarizes the required properties of the induced map F .

Proposition 6. *Let W be a thin set. Let λ be a supergrowing parameter; put $f = f_\lambda$. For every $\delta > 0$ there exist $M \in \mathbb{N}$ and a map F defined in W_M , with the following properties:*

- *F is constructed with appropriate iterates of the map f and for every rectangle $R_r^k \in Z_M$ $F|_{R_r^k} = f^n$ for some $n \in \mathbb{N}$.*

•

$$(3.2) \quad F(\Lambda_W \cap Y_M) \subset \Lambda_W \cap Y_M$$

- *For every rectangle $R_r^k \in Z_M$ the set $R_r^k \cap F^{-1}(W_M)$ can be covered by a family \mathcal{F}_r^k of disjoint subsets of R_r^k such that each set $Q \in \mathcal{F}_r^k$ is mapped by F bijectively onto its image $F(Q)$, which is contained in some rectangle $\hat{R}_s^m \in Z_M$. Moreover, the holomorphic branch of F^{-1} mapping $F(Q)$ back onto Q is well defined in a neighbourhood of the whole rectangle \hat{R}_s^m and*

$$(3.3) \quad \sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} < \frac{1}{2}$$

4. DEFINITION AND THE ESTIMATES FOR THE MAP F IN W_M^+

Let λ be a parameter satisfying the assumption of Proposition 6. We use the simplified notation $f = f_\lambda$. The map F is defined in W_M^+ simply as $F(x) = f(x)$. Thus, F is one-to-one, if restricted to any rectangle R_r^k .

Lemma 7. *For M large enough the inclusion*

$$F(\Lambda_W \cap Y_M^+) \subset \Lambda_W \cap Y_M$$

holds.

Proof. It is obvious that $f(\Lambda_W) \subset \Lambda_W$. So, $F(\Lambda_W \cap Y_M^+) \subset \Lambda_W$.

Let $z \in \Lambda_W \cap Y_M^+$. Since $f(z) \in W$, we have

$$K(|\operatorname{Re}(f(z))| + 1) > |f(z)| = |\lambda|e^{\operatorname{Re}(z)} \geq |\lambda|e^M.$$

where K is the constant coming from the definition of a thin set.

Thus,

$$|\operatorname{Re}(F(z))| = |\operatorname{Re}(f(z))| \geq \frac{1}{K}|\lambda|e^M - \frac{1}{K}$$

Thus, if M is large enough, we conclude that $|\operatorname{Re}(F(z))| \geq M$. □

Now, we turn to the proof of (3.3). More precisely, we shall estimate the sum in (3.3), corresponding to the rectangles $R_r^k \in Z_M$, for r positive (thus: larger than or equal to M). The estimate is straightforward and based on Lemma 8 below.

Lemma 8. Fix some $\delta \in (0, 1)$, and put $\delta' = \frac{\delta}{2}$. There exist \widehat{C} (independent of δ) and $M > 0$ (depending on δ) such that, for $r \geq M$, and every rectangle R_r^k

$$(4.1) \quad \sum_{\widehat{R}_s^l \cap f(R_r^k) \neq \emptyset} \sup_{w \in f^{-1}(\widehat{R}_s^l) \cap R_r^k} |f'(w)|^{-(1+\delta)} \leq \widehat{C} e^{-r\delta'}$$

where the summation runs over all rectangles $\widehat{R}_s^l \in Z_M$ intersecting $f(R_r^k)$.

Proof. We assume now that M is so large that $w(s) < s^{\delta'}$ for all $s \geq M$.

If $z \in R_r^k$, we have

$$(4.2) \quad |f'(z)| = |f(z)| = |\lambda| e^{\operatorname{Re}(z)} \geq |\lambda| e^r$$

The number of rectangles $\widehat{R}_s^l \in Z_M$ for which $\widehat{R}_s^l \cap f(R_r^k) \neq \emptyset$ can be estimated (very roughly) by

$$\sum_{M \leq s \leq |\lambda| e^{r+1}} w(s) \leq \sum_{M \leq s \leq |\lambda| e^{r+1}} s^{\delta'} \leq C e^{(1+\delta')(r+1)}.$$

Thus,

$$\sum_{\widehat{R}_s^l \cap f(R_r^k) \neq \emptyset} \sup_{w \in f^{-1}(\widehat{R}_s^l) \cap R_r^k} |f'(w)|^{-(1+\delta)} \leq C e^{(1+\delta')(r+1)} |\lambda|^{-(1+\delta)} e^{-r(1+\delta)} = \widehat{C} e^{-r\delta'}$$

where C and \widehat{C} are constants independent of M and δ (see Figure 1). \square

Assuming that M is so large that $\widehat{C} e^{-M\delta'} < \frac{1}{2}$ and denoting by \mathcal{F}_r^k the family of all sets of the form $Q = f^{-1}(\widehat{R}_s^l) \cap R_r^k$, we thus get the inequality (3.3):

$$(4.3) \quad \sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} \leq \widehat{C} e^{-r\delta'} \leq \frac{1}{2}$$

Remark. Note that, in this part of the proof we did not use the "super-growing" property. We used only the fact that the domain of the map is contained in $\operatorname{Re} z \geq M$ and that the set W is thin. It is worth to observe that above calculation is close in spirit to the argument contained in [Ka].

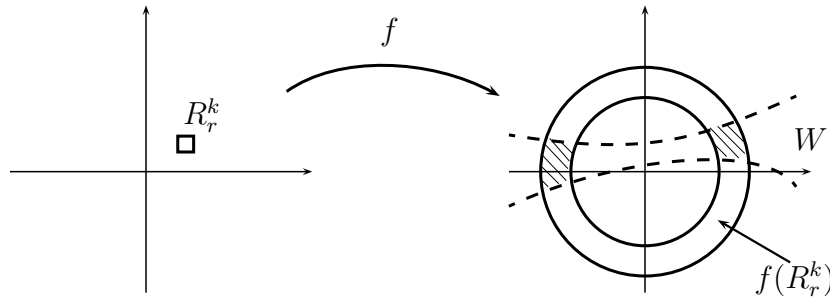


FIGURE 1. F in W_M^+

5. DEFINITION AND THE ESTIMATES FOR THE MAP F IN W_M^- .

We keep the assumption that λ is a supergrowing parameter and, as before, we abbreviate the notation writing $f := f_\lambda$.

We start with the following easy observation.

Proposition 9. *If the singular value 0 does not belong to the set Λ_W , then there exists $M > 0$ such that the set Λ_W is contained in the right half-plane $\operatorname{Re} z \geq -M$.*

Proof. If $0 \notin \Lambda_W$ then there exists $k \geq 0$ such that $f^k(0) \notin W$, and, consequently, there exists a ball $B(0, \eta)$ such that $f^k(z) \notin W$ for every $z \in B(0, \eta)$. Consequently, $B(0, \eta) \cap \Lambda_W = \emptyset$. Thus, if $\operatorname{Re} w < \log \eta - \log |\lambda|$ then $w \notin \Lambda_W$ (since $f(w) \in B(0, \eta)$). \square

Thus, the part of the proof contained in this section is void in the case when $0 \notin \Lambda_W$. The definition of the map F and the proof of (3.3) for r negative is more involved and it uses the "super-growing" condition.

The proof of the following technical lemma is straightforward and left to the reader.

Lemma 10. *Let λ be a super-growing parameter, $\alpha_n = \operatorname{Re} f_\lambda^n(0)$. Then for every $\varepsilon > 0$ there exists N such that for all $n > N$*

$$(5.1) \quad \alpha_1 + \alpha_2 + \cdots + \alpha_n < \varepsilon \alpha_{n+1}$$

The Koebe distortion theorem implies that for every univalent map f defined on some ball $B(z, r)$, the distortion of f' , restricted to the ball $B(z, \frac{r}{2})$ is bounded by some constant, independent of the map. This constant will be denoted by L . Recall the "super-growing" equivalent conditions (2.2), (2.3) and (2.4). Put

$$D = \frac{1}{4} \frac{c}{|\lambda|} \leq \frac{1}{4}$$

where the constant c comes from the super-growing conditions.

It follows from the condition (2.3) that there exists l_0 such that, for all $l \geq l_0$, the ball $B(\beta_l, 2D|\beta_l|)$ contains only one point of the trajectory of the point 0. Put

$$B_l := B(\beta_l, D|\beta_l|)$$

Thus, the inverse branches of f^l are well defined in B_l , with distortion bounded by L .

Let f_0^{-l} denote the branch of f^{-l} following the backward trajectory $\beta_l, \beta_{l-1}, \dots, \beta_0 = 0$; we put $\tilde{B}_l = f_0^{-l}(B_l)$. Then \tilde{B}_l is a topological disc containing the point 0. Since $|(f^l)'(0)| = |\beta_1| \cdots |\beta_l|$ and since the branch f_0^{-l} , sending β_l to 0, is also well defined on the ball twice larger $2 \cdot B_l = B(\beta_l, 2D|\beta_l|)$, the set \tilde{B}_l is contained in the ball

$$(5.2) \quad B\left(0, \frac{LD|\beta_l|}{|\beta_1| \cdots |\beta_{l-1}||\beta_l|}\right) = B\left(0, \frac{LD}{|\beta_1| \cdots |\beta_{l-1}|}\right)$$

and contains the ball

$$(5.3) \quad B\left(0, \frac{\frac{1}{4}D|\beta_l|}{|\beta_1| \cdots |\beta_{l-1}||\beta_l|}\right) = B\left(0, \frac{\frac{1}{4}D}{|\beta_1| \cdots |\beta_{l-1}|}\right).$$

Next, put $\tilde{G}_l = f^{-1}(\frac{1}{e}\tilde{B}_l)$. where $\frac{1}{e}\tilde{B}_l$ is the image of \tilde{B}_l under the rescaling $z \mapsto \frac{1}{e}z$. So, \tilde{G}_l is an unbounded set containing some left halfplane. Finally, put $G_l = \tilde{G}_l \cap W$ (see Figure 2).

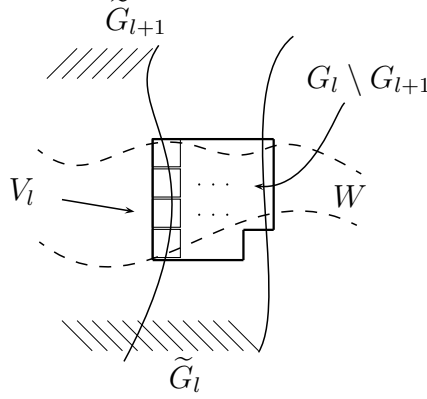


FIGURE 2. Definitions of sets V_l , \tilde{G}_l , etc.

Lemma 11. *For all l large enough, $\text{cl}\tilde{G}_{l+1} \subset \tilde{G}_l$. Consequently, $\text{cl}G_{l+1} \subset G_l$.*

Proof. Indeed, it follows from (5.2) and (5.3) that, for l large, $\text{cl}\tilde{B}_{l+1} \subset \tilde{B}_l$. \square

Finally, set V_l to be the union of all rectangles $R_r^k \in Z_M$ which intersect the set $G_l \setminus G_{l+1}$. Thus the sets V_l are not disjoint, but, for large l , one rectangle R_r^k may intersect two sets V_l at most (see Figure 2).

Given $l_0 \in \mathbb{N}$, we put $M = [\alpha_{l_0}] + 1$. Note that, by the above estimates, we have:

Lemma 12. *If l_0 is large enough then*

$$W_M^- \subset \bigcup_{l=l_0+1}^{\infty} V_l.$$

Proof. Using the fact that \tilde{B}_l contains the ball $B(0, \frac{\frac{1}{4}D}{|\beta_1| \cdots |\beta_{l-1}|})$, we conclude that the set \tilde{G}_l contains the left halfplane

$$(5.4) \quad \{\text{Re}(z) < -\log 4 - 1 + \log D - (\alpha_{l-2} + \cdots + \alpha_0) - l \log |\lambda|\}$$

It follows from (5.4) and Lemma 10 that, for large l , \tilde{G}_l contains the left halfplane

$$\{\text{Re} z \leq -\alpha_{l-1}\}.$$

Thus,

$$W_M^- = W \cap \{\operatorname{Re} z \leq -([\alpha_{l_0}] + 1)\} \subset W \cap \tilde{G}_{l_0+1} = G_{l_0+1} \subset \bigcup_{l=l_0+1}^{\infty} V_l.$$

□

Below, we define the map F , separately on each set V_l .

For $z \in V_l$ define $F(z) = f \circ f^l \circ f = f^{l+2}$. Note that the set \tilde{G}_l is mapped by f onto $\frac{1}{e}\tilde{B}_l$. Thus, f maps V_l into \tilde{B}_l and we have

$$V_l \xrightarrow{f} \tilde{B}_l \xrightarrow{f^l} B_l \xrightarrow{f} f(B_l).$$

Note that the map is neither onto nor injective. Obviously, the map $F|_{V_l}$ has a holomorphic extension to \mathbb{C} .

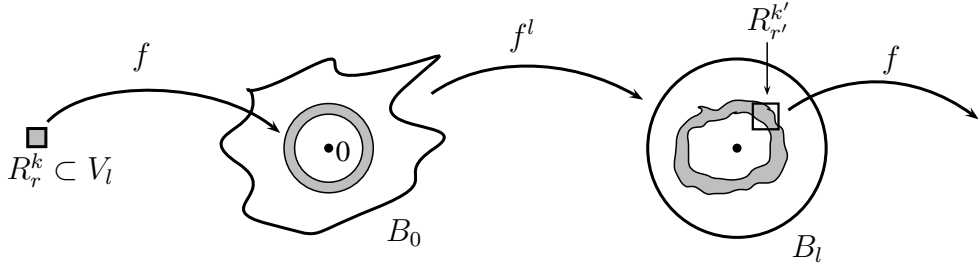


FIGURE 3. Defining F in W_M^-

The following lemma can be proved similarly as Lemma 7.

Lemma 13. *Put $M = [\alpha_{l_0}] + 1$. If l_0 is sufficiently large then, for all $l \geq l_0$,*

$$F(\Lambda_W \cap V_l) \subset \Lambda_W \cap Y_M.$$

Proof. It is obvious that $F(\Lambda_W \cap V_l) \subset \Lambda_W$. Next, $f^{l+1}(V_l) \subset B_l$, so taking $w \in \Lambda_W \cap V_l$, we have for $f^{l+1}(w) = z \in f^{l+1}(V_l)$:

$$\operatorname{Re}(z) \geq \operatorname{Re}\beta_l - D|\beta_l| \geq \alpha_l - \frac{1}{4}\alpha_l = \frac{3}{4}\alpha_l$$

Thus,

$$|F(w)| = |f(f^{l+1})(w)| \geq e^{\frac{1}{2}|\alpha_l|}$$

if l is large enough. As in the proof of Lemma 7 we conclude that

$$|\operatorname{Re} F(w)| \geq \frac{1}{K}|F(w)| - \frac{1}{K} \geq \frac{1}{K}e^{\frac{1}{2}\alpha_l} - \frac{1}{K} > [\alpha_l] + 1 \geq [\alpha_{l_0}] + 1$$

for all $l \geq l_0$, if l_0 is large enough. Therefore, $F(w) \in Y_M$.

□

Now, we turn to the proof of the estimate (3.3) (for negative r). Let us fix some $\delta \in (0, 1)$. Let, as in Section 4, $\delta' = \delta/2$. Let l_0 be so large that the statement of Lemma 8 holds with M replaced by $\frac{1}{2}[\alpha_{l_0}]$.

Let R_r^k be a rectangle in V_l . Then f^{l+1} maps R_r^k bijectively onto its image, contained in B_l , and for $z \in R_r^k$ we have:

(5.5)

$$|(f^{l+1})'(z)| = |f'(z)| |(f^l)'(f(z))| = |f(z)| |(f^l)'(f(z))| \geq \frac{\frac{1}{4e}D}{|\beta_1\beta_2\ldots\beta_l|} \cdot L^{-1}|\beta_1\beta_2\ldots\beta_l| = \frac{D}{4eL}$$

The whole set B_l is covered by $O(|\beta_l|^2)$ rectangles $R_{r'}^{k'}$ of the initial partition (3.1).

On each such rectangle $R_{r'}^{k'}$ the map f is a bijection onto its image. Thus, as in Section 4, there a family $\mathcal{F}_{r'}^{k'}$ of disjoint sets $Q_{r'}^{k'}$, such that the set $R_{r'}^{k'} \cap f^{-1}(W_M)$ can be written as a union of the sets $Q_{r'}^{k'} \in \mathcal{F}_{r'}^{k'}$. Each set $Q_{r'}^{k'}$ is defined as the intersection $f^{-1}(\widehat{R}_s^m) \cap R_{r'}^{k'}$ where \widehat{R}_s^m is some rectangle from the family Z_M . Obviously, the inverse map f^{-1} is well defined and holomorphic in a neighbourhood of every rectangle \widehat{R}_s^m .

Moreover, since the ball B_l is contained in the set $\{\operatorname{Re}(z) > \frac{1}{2}\alpha_l\}$, for each such rectangle $R_{r'}^{k'}$ Lemma 8 tells that

(5.6)

$$\sum_{Q_{r'}^{k'} \in \mathcal{F}_{r'}^{k'}} \sup_{z \in Q_{r'}^{k'}} |f'(z)|^{-(1+\delta)} < \hat{C} e^{-\delta' \cdot \frac{1}{2}\alpha_l}$$

Now, recall that f^{l+1} maps the rectangle R_r^k bijectively onto its image, contained in B_l , see Figure 3. Let $g = (f^{l+1}|_{R_r^k})^{-1}$ be the inverse map.

Taking all the preimages $g(Q)$, $Q \in \mathcal{F}_{r'}^{k'}$ over all rectangles $R_{r'}^{k'}$ intersecting B_l , we obtain the family \mathcal{F}_r^k of disjoint subsets of the rectangle R_r^k , such that the union $\bigcup_{Q \in \mathcal{F}_r^k} Q$ covers the whole set $R_r^k \cap F^{-1}(W_M)$ and each set $Q \in \mathcal{F}_r^k$ is mapped by $F = f^{l+2}$ bijectively onto its image, contained in some rectangle $\widehat{R}_s^m \in Z_M$.

Denote by \mathcal{G}_l the family of all rectangles $R_{r'}^{k'}$ intersecting the ball B_l . Using (5.5) and (5.6), we get the following estimate.

$$\begin{aligned} & \sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} \leq \\ (5.7) \quad & \leq \sup_{w \in R_r^k} |(f^{l+1})'(w)|^{-(1+\delta)} \cdot \sum_{R_{r'}^{k'} \in \mathcal{G}_l} \left(\sum_{Q_{r'}^{k'} \in \mathcal{F}_{r'}^{k'}} \sup_{z \in Q_{r'}^{k'}} |(f')(z)|^{-(1+\delta)} \right) \\ & \leq (4eLD^{-1})^{(1+\delta)} (\#\mathcal{G}_l) \cdot \hat{C} e^{-\delta' \cdot \frac{1}{2}\alpha_l} \end{aligned}$$

Since the cardinality $\#\mathcal{G}_l$ can be estimated by $\mathcal{O}(|\beta_l|^2)$, we can write

$$\sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} \leq C_1 |\beta_l|^2 e^{-\frac{1}{2}\delta'\alpha_l} = C_1 |\beta_l|^2 |\lambda|^{\frac{1}{2}\delta'} |\beta_{l+1}|^{-\frac{1}{2}\delta'}$$

where C_1 is another constant. Using the "super-growing" condition we easily conclude that the right-hand side of the above inequality can be estimated from above by

$$|\beta_{l+1}|^{-\frac{\delta'}{4}},$$

for all $l \geq l_0$, if l_0 is large enough. So, finally we get, for every rectangle R_r^k intersecting V_l ,

$$(5.8) \quad \sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} \leq |\beta_{l+1}|^{-\frac{\delta'}{4}} < \frac{1}{2}$$

This ends the proof of Proposition 6, with $M = [\alpha_{l_0}] + 1$, and F given by

$$(5.9) \quad F(z) = \begin{cases} f(z) & \text{for } z \in W_M^+ \\ f^{l+2}(z) & \text{for } z \in V_l, l \geq l_0 + 1 \end{cases}.$$

where l_0 is chosen large enough to satisfy all the estimates in the present Section (and, consequently, the estimate (5.8)). The condition (3.2) is satisfied because of Lemma 7, Lemma 12 and Lemma 13. The condition (3.3) is satisfied because of the estimates (4.3) and (5.8).

Remark. Recall that $W_M^- \subset \bigcup_{l=l_0}^\infty V_l$, so the map F is defined everywhere in W_M . However, since the sets V_l are not disjoint, there are rectangles R_r^k that are included in both V_l and V_{l+1} . In such case we define the map F on the rectangle R_r^k , choosing arbitrarily one of two possible ways.

6. ESTIMATES OF THE DIMENSION

In this section, we finish the proof of Theorem 4. Fix an arbitrary $\delta > 0$ and let M be so large that the statement of Proposition 6 holds true.

We shall define inductively, for every rectangle $R_r^k \in Z_M$, and for every n , a cover $(\mathcal{F}_r^k)^n$ of the set $\Lambda_W \cap R_r^k$, such that

$$(6.1) \quad \sum_{K \in (\mathcal{F}_r^k)^n} (\text{diam} K)^{1+\delta} < (2\pi + 1) \cdot \frac{1}{2^n}.$$

The cover $(\mathcal{F}_r^k)^1$ is defined simply as the family of sets \mathcal{F}_r^k described in Proposition 6. The condition (6.1) is satisfied since

$$\sum_{Q \in \mathcal{F}_r^k} (\text{diam} Q)^{1+\delta} \leq \left(\sum_{Q \in \mathcal{F}_r^k} \sup_{w \in Q} |F'(w)|^{-(1+\delta)} \right) \cdot (2\pi + 1) \leq \frac{1}{2} \cdot (2\pi + 1).$$

Assume that the covers $(\mathcal{F}_r^k)^{n-1}$ have been already defined for every rectangle $R_r^k \in Z_M$. So, fix some rectangle $R_r^k \in Z_M$, and, next, some set $Q_r^k \in \mathcal{F}_r^k$. It then follows from the construction that the set Q_r^k is mapped by F bijectively onto its image $F(Q_r^k)$, which is contained in some rectangle $\hat{R}_s^m \in Z_M$.

By the inductive assumption, for the rectangle $\hat{R}_s^m \in Z_M$ there is a cover $(\hat{\mathcal{F}}_s^m)^{n-1}$ of the set $\hat{R}_s^m \cap \Lambda_W$, such that

$$\sum_{\widehat{K} \in (\widehat{\mathcal{F}}_s^m)^{n-1}} (\text{diam}(\widehat{K}))^{1+\delta} < (2\pi + 1) \frac{1}{2^{n-1}}.$$

This allows us to define a cover \mathcal{Q}_r^k of the set $Q_r^k \cap \Lambda_W$ as the family of all sets of the form $F_*^{-1}(\widehat{K})$ where $\widehat{K} \in (\widehat{\mathcal{F}}_s^m)^{n-1}$ and F_*^{-1} is the inverse of the bijective map $F : Q_r^k \rightarrow F(Q_r^k)$. Obviously,

$$\sum_{\widehat{K} \in (\widehat{\mathcal{F}}_s^m)^{n-1}} \text{diam}(F_*^{-1}(\widehat{K}))^{1+\delta} \leq (2\pi + 1) \frac{1}{2^{n-1}} \cdot \sup_{w \in Q_r^k} |F'(w)|^{-(1+\delta)}$$

Finally, we define the family $(\mathcal{F}_r^k)^n$ as the union (over all sets $Q_r^k \in \mathcal{F}_r^k$) of all the covers \mathcal{Q}_r^k , described above. Since \mathcal{Q}_r^k is a cover of $Q_r^k \cap \Lambda_W$, we obtain, by taking the union, a cover of $R_r^k \cap \Lambda_W$.

Moreover, by (3.3),

$$(6.2) \quad \sum_{K \in (\mathcal{F}_r^k)^n} (\text{diam} K)^{1+\delta} \leq \sum_{Q_r^k \in \mathcal{F}_r^k} \sup_{w \in Q_r^k} |F'(w)|^{-(1+\delta)} \cdot (2\pi + 1) \frac{1}{2^{n-1}} \leq \frac{1}{2} \cdot (2\pi + 1) \frac{1}{2^{n-1}} = (2\pi + 1) \frac{1}{2^n}$$

This ends the proof of the inequality $\dim_H(\Lambda_W \cap R_r^k) \leq 1 + \delta$. The conclusion $\dim_H(\Lambda_W \cap Y_M) \leq 1 + \delta$ is immediate. Theorem 4 is proved.

7. APPLICATION-HAUSDORFF DIMENSION OF INDECOMPOSABLE CONTINUA

In this Section we apply Theorem 4 to show that the Hausdorff dimension of several indecomposable continua, described in Section 1, is equal to one.

The strategy is the following. Since every non-trivial continuum in the plane has Hausdorff dimension at least one, it is enough to prove the upper bound on the dimension. We shall use Theorem 4, and, more precisely, Corollary 5. In order to make use of Corollary 5, we shall check that our continuum, minus at most one point, is contained in the set $\Lambda_{W,ubd}$ for some thin set W . This has already been observed in several particular cases (see [DJ], [De2], and [FRS]) for the most general statement).

For the completeness, we outline the proof of Lemma 15 below. This is a particular case, needed in the present paper. We start with a standard fact. See e.g. [UZ], Lemma 2.2 for its proof.

Lemma 14. *Fix some parameter λ for which the singular trajectory $f_\lambda^n(0)$ escapes (i.e. $f_\lambda^n(z) \rightarrow \infty$). Fix some $R > 0$, and denote by F_R the set of points $z \in \mathbb{C}$ for which the whole trajectory $f_\lambda^n(z)$ remains in the closed ball $\overline{B}(0, R)$. Then the map $f_{\lambda|F_R}$ is expanding: there exist $c > 0$ and $\gamma > 1$ such that for every $z \in F_R$*

$$|(f_\lambda^n)'(z)| \geq c\gamma^n.$$

We now assume that the parameter λ is chosen so that the singular value 0 escapes sufficiently fast. More precisely, it follows from the classification of the escaping points ([SchZi] Corollary 6.9) that the fact that $f_\lambda^n(0)$ escapes to ∞ implies that there is a dynamic ray $g_{\underline{r}}$ such that $0 = g_{\underline{r}}(t_0)$ for some $t_0 \geq t_{\underline{r}}$. In Lemma 15 below, we shall assume additionally that $t_0 > t_{\underline{r}}$, i.e. that the point 0 is located on an (open) ray.

We need the following.

Lemma 15. *Fix some parameter λ , such that the singular trajectory $f_\lambda^n(0)$ escapes, and $0 = g_{\underline{r}}(t_0)$ for some $t_0 > t_{\underline{r}}$. Let $g_{\underline{s}} : (t_{\underline{s}}, \infty)$ be a dynamic ray. Denote by $G_{\underline{s}}$ the closure of the set $\{g_{\underline{s}}(t), t > t_{\underline{s}}\}$ in \mathbb{C} . Then at most one point in $G_{\underline{s}}$ has bounded trajectory.*

Proof. In the outline below we use notation from [SchZi] and [Rem]. To simplify the notation, we write below $g_{\underline{r}}$ to denote the arc $\{g_{\underline{r}}(t), t \geq t_0\}$. Put $f = f_\lambda$. Now, it is convenient to use a "dynamical coding", as proposed in [Rem], Section 2. The preimage $f^{-1}(g_{\underline{r}})$ is a union of countably many curves $g_{k\underline{r}}$, which cut the plane, defining countably many open strips S_k , where S_k is bounded by $g_{k\underline{r}}$ and $g_{(k+1)\underline{r}}$. Denote also by \overline{S}_k the closure of S_k .

Each strip S_k is mapped by f univalently onto $\mathbb{C} \setminus g_{\underline{r}}$. Denote by f_k^{-1} the inverse map: $f_k^{-1} : \mathbb{C} \setminus g_{\underline{r}} \rightarrow S_k$.

Let $g_{\underline{s}}$ be an arbitrary dynamic ray. It is easy to check that the ray $g_{\underline{s}}$ does not intersect the ray $g_{k\underline{r}}$ unless $g_{\underline{s}} = g_{k\underline{r}}$. Thus, all the points on the ray $g_{\underline{s}}$ share the same "dynamic address": there is a sequence $\tilde{s} = \tilde{s}_0, \tilde{s}_1, \dots$ such that for every point $z \in g_{\underline{s}}$ and every n we have $f^n(z) \in S_{\tilde{s}_n}$. Obviously, for every $z \in G_{\underline{s}}$ we have $f^n(z) \in \overline{S}_{\tilde{s}_n}$.

Now, let us fix some $R > 0$, and consider the set F_R defined in Lemma 14. Since all points in the rays $g_{k\underline{r}}$ escape to ∞ , and the trajectory of points $z \in F_R$ remains bounded, there exists $\delta > 0$ such that the trajectory of every point $z \in F_R$ is δ -separated from the boundary curves of all strips S_k , and, by the same reason, from the curves $f^k(g_{\underline{r}})$. (Here we use the fact that $f^k(z) \rightarrow \infty$ uniformly on the curve $g_{\underline{r}} := g_{\underline{r}}([t_0, \infty))$, so, actually, only finitely many of curves $f^k(g_{\underline{r}})$ intersect the ball $\overline{B}(0, R)$.)

This easily implies the following: there exists a constant $L > 0$ such that, for arbitrary two points $z, w \in F_R$, belonging to the same strip, there exist a topological disc $U = U_{z,w}$ on which each composition

$$f_{k_0}^{-1} \circ f_{k_1}^{-1} \circ \dots \circ f_{k_{n-1}}^{-1}$$

is well defined, with distortion bounded by L .

Let $x, y \in F_R$. If $x, y \in G_{\underline{s}}$, then they have the same dynamic code $\tilde{s}(x) = \tilde{s}(y) = \tilde{s}_0, \tilde{s}_1, \dots, \tilde{s}_{n-1}, \tilde{s}_n, \dots$. Put $z = f^n(x)$, $w = f^n(y)$ and take the inverse branch following our coding

$$f_*^{-n} = f_{\tilde{s}_0}^{-1} \circ f_{\tilde{s}_1}^{-1} \circ \dots \circ f_{\tilde{s}_{n-1}}^{-1}.$$

Then $f_*^{-n}(z) = x$ and $f_*^{-n}(w) = y$ (because $\tilde{s}(x) = \tilde{s}(y)$). Expanding property (Lemma 14) together with the bounded distortion property give

$$|x - y| = |f_*^{-n}(z) - f_*^{-n}(w)| \leq \frac{L \cdot 2R}{c} \gamma^{-n}.$$

Since n can be taken arbitrarily large, we get $x = y$. □

Note that, actually, we proved in Lemma 15 the following fact:

Lemma 16. *Under the assumption and notation of Lemma 15, let $\tilde{s} = (\tilde{s}_0, \tilde{s}_1, \tilde{s}_2 \dots)$ be a dynamic address and let $H_{\tilde{s}}$ be the set of points $z \in \mathbb{C}$ such that, for every $n \in \mathbb{N}$, $f_\lambda^n(z) \in \overline{S}_{\tilde{s}_n}$. Then at most one point in $H_{\tilde{s}}$ has bounded trajectory.*

We are ready to formulate the following corollaries.

Theorem 17. *Let $f(z) = \exp(z)$ and let Λ be the indecomposable continuum described in [De2]:*

$$\Lambda = \{z \in \mathbb{R} : \forall n \geq 0 \operatorname{Im}(f^n z) \in [0, \pi]\}$$

Then $\dim_H(\Lambda) = 1$.

Proof. This is an immediate consequence of Theorem 4 and Lemma 16: Putting

$$W = \{z : \operatorname{Im}(z) \in [0, \pi]\}$$

we see immediately that Λ is contained in the set of points whose trajectory remains in the closed dynamic strip $\overline{S}_0 = \mathbb{R} \times [0, 2\pi]$. (Note, besides, that points from $S_0 \setminus W$ will leave S_0 immediately.) Thus, using Lemma 16 we see that $\operatorname{card}(\Lambda \setminus \Lambda_{W,ubd}) \leq 1$. Therefore, by Theorem 4, $\dim_H(\Lambda) \leq 1$. Since Λ is a non-trivial continuum, $\dim_H(\Lambda) \geq 1$. This gives the required equality. □

Remark 3. The inclusion $\operatorname{card}(\Lambda \setminus \Lambda_{W,ubd}) \leq 1$ can be also deduced from the detailed description of ω -limit sets of the points in Λ , provided in [De2].

Now, let us consider a more general situation: Assume that the parameter λ satisfies $f_\lambda^n(0) \rightarrow \infty$. In particular, this implies that the singular value 0 is on a dynamic ray or is the landing point of such a ray. Denote this ray, as in the proof of Lemma 15, by $g_{\underline{r}} : (t_{\underline{r}}, \infty) \rightarrow \mathbb{C}$. For such maps, L. Rempe provides the construction of uncountably many dynamically defined indecomposable continua. Each such a continuum is defined as the accumulation set of some dynamic ray (see [Rem], Theorem 1.2). Namely, one considers the set R_1 of external addresses of the following form:

$$\underline{s} = s(n_1, n_2, n_3, \dots) := T_1 r_0 r_1 \dots r_{n_1-1} T_{n_1} r_0 r_1 \dots r_{n_2-1} T_{n_2} r_0 r_1 \dots r_{n_3-1} T_{n_3} \dots$$

where $T_n := 2 + \max_{k \leq n} r_k$. A suitably chosen, uncountable subset $R \subset R_1$ has the required property: for every ray $\underline{s} \in R$ the accumulation set of the ray $g_{\underline{s}}$ is an indecomposable continuum.

Before formulating the result about the continua described in [Rem], we note the following auxiliary fact.

Lemma 18. *Let $\lambda \in \mathbb{C} \setminus \{0\}$ is chosen so that the trajectory $f_\lambda^n(0)$ escapes to ∞ .*

Let $g_{\underline{u}} : (t_{\underline{u}}, \infty) \rightarrow \mathbb{C}$ be a dynamic ray such that the sequence u_n is bounded. If the ray $g_{\underline{u}}$ "lands", i.e. if there exists a finite limit

$$\lim_{t \searrow t_{\underline{u}}} g_{\underline{u}}(t) = a \in \mathbb{C}$$

then the trajectory $f_\lambda^n(a)$ does not escape to ∞ .

Proof. Let $K = \sup_n \{|u_n|\}$. As mentioned above, since the point 0 escapes, there is an external ray $g_{\underline{t}} : (t_{\underline{t}}, \infty) \rightarrow \mathbb{C}$ such that 0 is either its landing point or it is contained in $g_{\underline{t}}(t_{\underline{t}}, \infty)$. Then, as in the proof of Lemma 15, and using its notation, we consider the dynamic coding defined with use of the curves $g_{k\underline{t}}$. Since the rays do not intersect, every image of the ray $f_\lambda^n(g_{\underline{u}}([t_{\underline{u}}, \infty)))$, is contained in some strip $S_{k(n)}$, and it is easy to see, using the behaviour of the rays at infinity (see Prop. 3.4 in [SchZi]) that $|k(n)| < K''$ for some $K'' \geq K$. Moreover, there exist $M' > 0$ and $K' \geq K''$ such that

$$\bigcup_{|k| < K'} S_k \cap \{\operatorname{Re} z > M\} \subset \bigcup_{|k| < K'} P_k.$$

Denote by $\Sigma_{K'}$ the set of all infinite sequences $\underline{w} = (w_n)_{n=0}^\infty$, with integer entries, and such that $|w_n| < K'$ for all n . Every point $z \in \mathbb{C}$ has its "geometric address" $\underline{s}(z)$ defined by $s_i(z) = k$ if $f_\lambda^n(z) \in P_k$. According to [DK], Section 3 (see also [SchZi], Prop. 3.4), there exists $M > 0$ depending on K' and λ such that the set of "directly escaping points":

$$E_{M,K'} = \{z \in \mathbb{C} : \operatorname{Re} f_\lambda^n(z) \geq M \text{ for all } n \geq 0 \text{ and } \underline{s}(z) \in \Sigma_{K'}\}$$

is a union of "tails of rays" $g_{\underline{s}}(t)$, $\underline{s} \in \Sigma_{K'}$. Each tail is a curve to ∞ , and, actually, a graph of a function defined in $\{x \geq M\}$, and for every $z \in g_{\underline{s}}([M, \infty)$ we have $\underline{s}(z) = \underline{s}$.

Now, let a be the landing point of the tail $g_{\underline{u}}$, and assume that $f_\lambda^n(a) \rightarrow \infty$. Then there exists n_0 such that for $n \geq n_0$ $\operatorname{Re} f_\lambda^n(a) > M + 1$. Consequently, $f_\lambda^{n_0}(a) \in E_{M,K'}$ and, since $\operatorname{Re} f^{n_0}(a) > M + 1$, $f^{n_0}(a)$ is in a tail of some ray (thus, it is located in an open ray). This is a contradiction since f_λ maps ends of rays to ends of rays. \square

Theorem 19. *Assume that the parameter λ satisfies $f_\lambda^n(0) \rightarrow \infty$. Assume additionally that $\operatorname{Im}(f_\lambda^n(0))$ remains bounded. Let Λ be the indecomposable continuum constructed in [Rem]. Then $\dim_H(\Lambda) = 1$.*

Proof of Theorem 19. As mentioned in Section 2, it is easy to see that the assumptions of the Theorem imply that the parameter λ satisfies the super-growing condition (2.2). Since $0 \in I(f_\lambda)$, it follows from the classification of escaping points ([SchZi]) that 0 belongs to some dynamic ray $g_{\underline{t}}(t_{\underline{t}}, \infty) \rightarrow \mathbb{C}$, or it is a landing point of a ray. Lemma 18 allows us to exclude this second possibility. Therefore, $0 = g_{\underline{t}}(t)$ for some $t > t_{\underline{t}}$. Open dynamic rays are smooth curves [V], see also [FS] for the estimates of the second derivative of the parametrization $t \mapsto g_{\underline{t}}(t)$, $t \in (t_{\underline{t}}, \infty)$. In particular, the ray $g_{\underline{t}}(t_{\underline{t}}, \infty)$ has a tangent vector at 0. Let us consider, as in the proof of Lemma 15, the rays $g_{k\underline{t}}$. Then the existence of a tangent vector of $g_{\underline{t}}$ at 0 guarantees that for every $k \in \mathbb{Z}$ there exists a finite limit

$$\lim_{\operatorname{Re} z \rightarrow -\infty, z \in g_{k\underline{t}}} \operatorname{Im}(z) = A_k$$

Moreover, it follows from the general bounds for the parametrization of hairs (see [SchZi], Prop 3.4) that there exists a finite limit

$$\lim_{\operatorname{Re} z \rightarrow +\infty, z \in g_{k\underline{t}}} \operatorname{Im} z = B_k = 2\pi k - \operatorname{Arg} \lambda$$

Obviously, $A_{k+1} = A_k + 2\pi$, $B_{k+1} = B_k + 2\pi$.

Now, let \underline{s} be an arbitrary external address with bounded entries, and let $g_{\underline{s}} : (t_{\underline{s}}, \infty) \rightarrow \mathbb{C}$ be the corresponding external ray. Put $G_{\underline{s}} = \overline{g_{\underline{s}}}$. As in the proof of Lemma 15 we notice that $G_{\underline{s}}$ is contained in the closure of some dynamic strip S_k bounded by the curves $g_{k\underline{r}}$ and $g_{(k+1)\underline{r}}$.

Moreover, again by the abovementioned Prop.3.4 in [SchZi] we know that for every dynamic ray $g_{\underline{s}}$

$$\lim_{t \rightarrow +\infty} \text{Im} g_{\underline{s}}(t) = 2\pi s_1 - \text{Arg} \lambda.$$

It now easily follows from two above observations that for every bounded sequence $g_{\underline{s}}$ there exists $K > 0$ such that

$$\bigcup_{n \geq 0} f_{\lambda}^n(G_{\underline{s}}) \subset W = \{z \in \mathbb{C} : |\text{Im} z| \leq K\}$$

Thus,

$$\Lambda \subset G_{\underline{s}} \subset \Lambda_W.$$

By Lemma 15 we have $\text{card}(G_{\underline{s}} \setminus \Lambda_{W, \text{ubd}}) \leq 1$. Therefore, using Theorem 4, we conclude that $\dim_H(\Lambda) \leq 1$, and, again, since Λ is a non-trivial continuum, $\dim_H(\Lambda) = 1$. \square

As a corollary we have

Corollary 20. *If Λ is an indecomposable continuum constructed, for the map $f_1(z) = \exp(z)$, in [DJ] then $\dim_H(\Lambda) = 1$.*

8. HAUSDORFF MEASURE OF THE CONTINUA

As a complement of our result let us note the following general remark:

Proposition 21. *One-dimensional Hausdorff measure of an indecomposable continuum in the plane is not σ -finite.*

To prove Proposition 21 we will need to state some facts about indecomposable continua. See [K] for definitions and properties. Let us denote the continuum by \mathcal{C} .

Definition 22. For a point p define the set K_p as a union of all proper subcontinua containing p , in other words $K_p = \{x : \exists \text{ a proper subcontinuum } S \text{ of } \mathcal{C} \text{ containing } p \text{ and } x\}$. The set K_p is called the *composant* of the point p .

The following facts are easy to verify cf. [K], Section 48.

Fact 1. *Every composant of an indecomposable metric continuum is an F_{σ} set of the first category.*

Fact 2. *In an indecomposable metric continuum every composant is a dense subset of \mathcal{C} .*

Fact 3. *An indecomposable metric continuum is a union of uncountably many disjoint composants.*

The next fact follows immediately from the definition of Hausdorff measure.

Fact 4. *A connected measurable set $A \subset \mathbb{R}^2$ with $\text{diam}(A) > 0$ has positive 1-dimensional Hausdorff measure $\mathcal{H}_1(A) > 0$.*

The above facts show that an indecomposable continuum consists of uncountably many disjoint sets of positive measure \mathcal{H}_1 . This proves that the whole set cannot be σ -finite with respect to \mathcal{H}_1 . The formal proof follows.

Proof of Proposition 21. Assume the opposite: the continuum \mathcal{C} can be represented as a union of countably many disjoint measurable sets of finite measure:

$$\mathcal{C} = \bigcup_{i=1}^{\infty} C_i, \quad C_i \cap C_j = \emptyset \text{ (for } i \neq j \text{) and } \mathcal{H}_1(C_i) < +\infty.$$

The set \mathcal{C} is also a union of disjoint composants of some points: $\mathcal{C} = \bigcup_{p \in P} K_p$, where $\text{card}(P) > \aleph_0$. Define the sets $K_p^i = K_p \cap C_i$.

The set \mathcal{C} has a positive diameter a . Since every K_p is dense in \mathcal{C} , the diameter of K_p is bigger than $\frac{1}{2}a$. Using Fact 4 we see that $\mathcal{H}_1(K_p) > 0$.

As $K_p = \bigcup_i K_p^i$ this means that for every p at least one K_p^i has positive measure $\mathcal{H}_1(K_p^i) > 0$. Let us denote these chosen sets by \widehat{K}_p .

Define $A_n = \{p \in P : \mathcal{H}_1(\widehat{K}_p) > \frac{1}{n}\}$. The set P is not countable so there exists $A_n = A_{n_0}$ containing uncountably many p 's. We denote them by \widehat{p} .

Finally, since there are countably many sets C_i , one of them contains infinitely many of $\widehat{K}_{\widehat{p}}$'s. Choosing a countable subset $(\widehat{p}_k)_{k=1}^{\infty}$, such that $\widehat{K}_{\widehat{p}_k} \subset C_i$, we can write

$$\mathcal{H}_1(C_i) \geq \mathcal{H}_1\left(\bigcup_{k=1}^{\infty} \widehat{K}_{\widehat{p}_k}\right) = \sum_{k=1}^{\infty} \mathcal{H}_1\left(\widehat{K}_{\widehat{p}_k}\right) = \infty$$

thus giving a contradiction. □

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